NOTE

Investigation of all Ricci semi-symmetric and all conformally semi-symmetric spacetimes

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Abstract.

We find all Ricci semi-symmetric as well as all conformally semi-symmetric spacetimes. Neither of these properties implies the other. We verify that only conformally flat spacetimes can be Ricci semi-symmetric without being conformally semi-symmetric and show that only vacuum spacetimes and spacetimes with just a Λ-term can be Ricci semi-symmetric without being conformally semi-symmetric.

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1. Introduction

Semi-symmetric spaces where introduced by Cartan [3] and are characterized by the curvature condition

$$\nabla_{[a}\nabla_{b]}R_{cdef} = 0, \tag{1}$$

where R_{cdef} denotes the Riemann tensor and round and square brackets enclosing indices indicate symmetrization and antisymmetrization, respectively.

A semi-Riemannian manifold is said to be *conformally semi-symmetric* if the Weyl tensor C_{abcd} satisfies

$$\nabla_{[a}\nabla_{b]}C_{cdef} = 0; \tag{2}$$

and *Ricci semi-symmetric* if the Ricci tensor satisfies

$$\nabla_{[a}\nabla_{b]}R_{cd} = 0. \tag{3}$$

The geometrical properties of semi-symmetric spaces where discussed in [11, 6, 8, 9, 4].

In this paper we find all semi-symmetric, Ricci semi-symmetric and all conformally semi-symmetric spacetimes. Eriksson and Senovilla [5] found all such non-conformally flat spacetimes.

It is an advantage to instead of tensors rather to use the Weyl spinor Ψ_{ABCD}, the curvature scalar spinor Λ, and the spinor $\Phi_{ABA'B'}$ for the tracefree part of the Ricci tensor. We also use the spinor $\mathrm{X}_{ABCD} = \Psi_{ABCD} + \Lambda\,(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC})$. The spinor

commutator $\Box_{AB}$ operating on spinors with one index is $\Box_{AB}\,\kappa_C = -\mathrm{X}_{ABC}{}^E\kappa_E$ and $\Box_{AB}\,\tau_{C'} = -\Phi_{ABC'}{}^{E'}\tau_{E'}$ [7].

In spinors (2) is equivalent to $\Box_{AB}\Psi_{CDEF} = 0$ and $\Box_{A'B'}\Psi_{CDEF} = 0$, or

$$\mathrm{X}_{AB(C}{}^G\Psi_{DEF)G} = 0, \tag{4}$$

$$\Phi_{A'B'(C}{}^G\Psi_{DEF)G} = 0. \tag{5}$$

2. Conditions for semi-symmetry

Calculation of the components of (4) in terms of Ψ_{ABCD} and Λ shows show that it not will have 15 independent components but rather just 5 components and (4) can be replaced by its contraction over BC or

$$\Psi^{GH}{}_{(AD}\Psi_{EF)GH} - 2\Lambda\Psi_{ADEF} = 0. \tag{6}$$

We observe that the spinor commutator $\Box_{AB}$ yields 0 if operating on a scalar as the curvature scalar Λ, therefore only its effect on $\Phi_{ABA'B'}$, i.e. on the tracefree part of the Ricci tensor has to be considered. The second condition for Ricci semi-symmetry (5) corresponds to spinor equations

$$\Box_{AB}\Phi_{CDC'D'} = -2\mathrm{X}_{AB(C}{}^E\Phi_{D)EC'D'} - 2\Phi_{AB(C'}{}^{E'}\Phi_{D')E'CD}, \tag{7}$$

$$\Box_{A'B'}\Phi_{CDC'D'} = -2\overline{\mathrm{X}}_{A'B'(C'}{}^{E'}\Phi_{D')E'CD} - 2\Phi_{A'B'(C}{}^E\Phi_{D)EC'D'}. \tag{8}$$

The condition (8) is however just the complex conjugate of (7) and will not give any additional conditions for Ricci semi-symmetry.

Equation (7) has three groups of two symmetric indices but will in fact not have 27 independent components as $\Box^{AB}\Phi_{ABC'D'} = 0$, it can be replaced by the fully symmetric spinors

$$\Box_{(AB}\Phi_{CD)C'D'} = -2\Psi_{(ABC}{}^E\Phi_{D)EC'D'} \tag{9}$$

$$\Box_{(A}{}^F\Phi_{C)FC'D'} = 4\Lambda\Phi_{ACC'D'} - \Psi^{EF}{}_{AC}\Phi_{EFC'D'} - 2\Phi^E{}_A{}^{F'}{}_{(C'}\Phi_{D')F'CE} \tag{10}$$

with 15 and 9 components respectively.

The various spacetimes are for simplicity studied in a frame where first Ψ_{ABCD} has been brought to a standard form depending on its Petrov type. Thereafter $\Phi_{ABA'B'}$ is brought to standard form depending on its Segre type [1, 10].

All of the above formulas have been implemented in CLASSI [2].

3. Main Results

The equations for Ricci semi-symmetric spacetimes (9) and (10) and Petrov Types **I**, **II** or **III** in standard frame requires that $\Phi_{ABA'B'} = 0$, while spacetimes with a Λ-term only or vacuum ($\Lambda = 0$) are Ricci semi-symmetric.

For Petrov type **D** spacetimes (with only Ψ_2 nonzero) the conditions both for conformal and for Ricci semi-symmetry require all components of $\Phi_{ABA'B'}$ to be zero except for $\Phi_{11'}$. The conditions for conformal semi-symmetry reduces to $2\Lambda+\Psi_2 = 0$ and

for Ricci semi-symmetry to $\Phi_{11'}(2\Lambda + \Psi_2) = 0$. Spacetimes of Segre type A1[(11)(1,1)] are therefore both Ricci and conformally semi-symmetric if they conform to $\Lambda = -\frac{1}{2}\Psi_2$ with arbitrary $\Phi_{11'}$. Spacetimes with Λ-term only are semi-symmetric if once again $\Lambda = -\frac{1}{2}\Psi_2$ but only Ricci semi-symmetric for other relations between Ψ_2 and Λ.

For Petrov type **N** spacetimes (with only Ψ_4 nonzero) the conditions for Ricci semi-symmetry as well as for conformal sem-symmetry require all components of $\Phi_{ABA'B'}$ to be zero except for $\Phi_{22'}$. The conditions here reduces to $\Lambda\Psi_4 = 0$ for conformal semi-symmetry and $\Lambda\Phi_{11'} = 0$ for Ricci semi-symmetry. They are therefore both Ricci and conformally semi-symmetric for arbitrary Ψ_4 and $\Phi_{22'}$ as long as $\Lambda = 0$. Once again all spacetimes with Λ-term only are Ricci semi-symmetric.

3.1. Conformally flat spacetimes

This is the case not treated by Eriksson and Senovilla [5].

All conformally flat (Petrov type **0**) spacetimes are conformally semi-symmetric.

Calculations with CLASSI shows that all conformally flat spacetimes of Segre type A1[(11)(1,1)] (only $\Phi_{11'}$ nonzero) with $\Lambda = 0$ are also Ricci semi-symmetric.

For conformally flat perfect fluids (Segre type A1[(111),1)], $\frac{1}{2}\Phi_{00'} = \Phi_{11'} = \frac{1}{2}\Phi_{22'}$) as well as for tachyons (Segre type A1[1(11,1)], $-\frac{1}{2}\Phi_{00'} = \Phi_{11'} = -\frac{1}{2}\Phi_{22'}$) the condition for Ricci semi-symmetry reduces to $\Lambda = \Phi_{11'}$. So perfect fluids with $\Lambda = \frac{1}{2}\Phi_{00'} = \Phi_{11'} = \frac{1}{2}\Phi_{22'}$ as well as tachyons with $\Lambda = -\frac{1}{2}\Phi_{00'} = \Phi_{11'} = -\frac{1}{2}\Phi_{22'}$ are also Ricci semi-symmetric.

All spacetimes of Segre type A3[(11,2)] with $\Lambda = 0$ (only $\Phi_{22'}$ nonzero) are also Ricci semi-symmetric.

All conformally flat spacetimes with Λ-term only, including flat spacetimes, are Ricci semi-symmetric.

For all other Segre types the conditions for Ricci semi-symmetry are not satisfied.

4. Summary

All conformally flat spacetimes are conformally semi-symmetric, all spacetimes with Λ-term only are Ricci semi-symmetric.

Semi-symmetric spacetimes are flat spacetimes, Λ-term and Segre A1[(11)(1,1)] spacetimes of Petrov type **0** and of type **D** with $\Lambda = -\frac{1}{2}\Psi_2$, Segre type A3[(11,2)] with $\Lambda = 0$ of Petrov types **N** and **0**, as well as conformally flat perfect fluids and tachyons with $\Lambda = \Phi_{11'}$. For a table see appendix A.

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Appendix A

Segre type Petrov type	**I**	**II**	**III**	**D**	**N**	**0**
Λ-term A1[(111,1)] or vacuum	Ric s-s	Ric s-s	Ric s-s	Ric s-s	Ric s-s	semi-sym
Λ-term, $\Lambda = -\frac{1}{2}\Psi_2$	Ric s-s	Ric s-s	$\nexists$	semi-sym	$\nexists$	$\nexists$
A1[(11)(1,1)], $\Lambda = -\frac{1}{2}\Psi_2$	-	-	$\nexists$	semi-sym	$\nexists$	$\nexists$
A1[(11)(1,1)]	-	-	-	see above	-	semi-sym
A3[(11,2)], $\Lambda = 0$	-	-	-	-	semi-sym	semi-sym
A1[(111),1] perfect fluid, $\Lambda = \frac{1}{2}\Phi_{00'} = \Phi_{11'} = \frac{1}{2}\Phi_{22'}$	-	-	-	-	-	semi-sym
A1[1(11,1)] tachyon fluid, $\Lambda = -\frac{1}{2}\Phi_{00'} = \Phi_{11'} = -\frac{1}{2}\Phi_{22'}$	-	-	-	-	-	semi-sym
All other Ricci tensors	-	-	-	-	-	conf s-s

Table 1. Relations between Petrov type, Segre type and conformal semi-symmetry (conf s-s), Ricci semi-symmetry (Ric s-s) and semi-symmetry. A hyphen (-) indicates neither conformal nor Ricci semi-symmetry.

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